

Concircular π -Vector Fields and Special Finsler Spaces*

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ABSTRACT

The aim of the present paper is to investigate intrinsically the notion of a concircular π -vector field in Finsler geometry. This generalizes the concept of a concircular vector field in Riemannian geometry and the concept of concurrent vector field in Finsler geometry. Some properties of concircular π -vector fields are obtained. Different types of recurrence are discussed. The effect of the existence of a concircular π -vector field on some important special Finsler spaces is investigated. Almost all results obtained in this work are formulated in a coordinate-free form.

Keywords: Finsler Manifold; Cartan Connection; Concurrent π -Vector Field; Concircular π -Vector Field; Special Finsler Space, Recurrent Finsler Space

1. Introduction

The concept of a concurrent vector field in Riemannian geometry had been introduced and investigated by K. Yano [1]. Concurrent vector fields in Finsler geometry had been studied *locally* by S. Tachibana [2], M. Matsumoto and K. Eguchi [3]. In [4], we investigated *intrinsically* concurrent vector fields in Finsler geometry. On the other hand, the notion of a concircular vector field in Riemannian geometry has been studied by Adat and Miyazawa [5]. Concircular vector fields in Finsler geometry have been studied *locally* by Prasad *et al.* [6].

In this paper, we introduce and investigate *intrinsically* the notion of a concircular π -vector field in Finsler geometry, which generalizes the concept of a concircular vector field in Riemannian geometry and the concept of a concurrent vector field in Finsler geometry. Some properties of concircular π -vector fields are obtained. These properties, in turn, play a key role in obtaining other interesting results. Different types of recurrence are discussed. The effect of the existence of a concircular π -vector field on some important special Finsler spaces is investigated: Berwald, Landesberg, C -reducible, semi- C -reducible, quasi- C -reducible, C_2 -like, S_3 -like, P -reducible, P_2 -like, h -isotropic, T^h -recurrent, T^v -recurrent, etc.

Global formulation of different aspects of Finsler geometry may help better understand these aspects without being trapped into the complications of indices. This is one of the motivations of the present work, where almost all results obtained are formulated in a coordi-

nate-free form.

2. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [7] and [8]. We shall use the same notations of [7].

In what follows, we denote by $\pi: TM \rightarrow M$ the tangent bundle to M , $\mathfrak{F}(TM)$ the algebra of C^∞ functions on TM , $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$ -module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π -vector fields and will be denoted by barred letters \bar{X} . The tensor fields on $\pi^{-1}(TM)$ will be called π -tensor fields. The fundamental π -vector field is the π -vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all $u \in TM$.

We have the following short exact sequence of vector bundles

$$0 \rightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \rightarrow 0,$$

with the well known definitions of the bundle morphisms ρ and γ . The vector space $V_u(TM) = \{X \in T_u(TM) : d\pi(X) = 0\}$ is the vertical space to M at $u \in TM$.

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map $K: T(TM) \rightarrow \pi^{-1}(TM): X \mapsto D_X \bar{\eta}$, called the connection map of D . The vector space $H_u(TM) = \{X \in T_u(TM) : K(X) = 0\}$ is the horizontal space to M at u . The connection D is said to be

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regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM), \forall u \in TM.$$

If M is endowed with a regular connection, then the vector bundle maps $\gamma, \rho|_{H(TM)}$ and $K|_{V(TM)}$ are vector bundle isomorphisms. The map $\beta := (\rho|_{H(TM)})^{-1}$ will be called the horizontal map of the connection D . We have $K \circ \gamma = id_{\pi^{-1}(TM)}$.

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of D , denoted by Q and T respectively, are defined by

$$Q(\bar{X}, \bar{Y}) = T(\beta\bar{X}, \beta\bar{Y}), T(\bar{X}, \bar{Y}) = T(\gamma\bar{X}, \beta\bar{Y}), \\ \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)),$$

where T is the (classical) torsion tensor field associated with D .

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of D , denoted by R, P and S respectively, are defined by

$$R(\bar{X}, \bar{Y})\bar{Z} = K(\beta\bar{X}, \beta\bar{Y})\bar{Z}, P(\bar{X}, \bar{Y})\bar{Z} = K(\beta\bar{X}, \gamma\bar{Y})\bar{Z}, \\ S(\bar{X}, \bar{Y})\bar{Z} = K(\gamma\bar{X}, \gamma\bar{Y})\bar{Z},$$

where K is the (classical) curvature tensor field associated with D .

The contracted curvature tensors of D , denoted by \hat{R}, \hat{P} and \hat{S} respectively, known also as the (v)h-, (v)hv- and (v)v-torsion tensors, are defined by

$$\hat{R}(\bar{X}, \bar{Y}) = R(\bar{X}, \bar{Y})\bar{\eta}, \hat{P}(\bar{X}, \bar{Y}) = P(\bar{X}, \bar{Y})\bar{\eta}, \\ \hat{S}(\bar{X}, \bar{Y}) = S(\bar{X}, \bar{Y})\bar{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) := g(R(\bar{X}, \bar{Y})\bar{Z}, \bar{W}), \dots, \\ S(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) := g(S(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) \quad (1)$$

The following theorem guarantees the existence and uniqueness of the Cartan connection on the pullback bundle.

Theorem 2.1. [9] *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L . There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

- a) ∇ is metric: $\nabla g = 0$;
- b) The (h)h-torsion of ∇ vanishes: $Q = 0$;
- c) The (h)hv-torsion T of ∇ satisfies:

$$g(T(\bar{X}, \bar{Y}), \bar{Z}) = g(T(\bar{X}, \bar{Z}), \bar{Y}).$$

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L) .

One can show that the (h)hv-torsion of the Cartan

connection is symmetric and has the property that $T(\bar{X}, \bar{\eta}) = 0$ for all $\bar{X} \in \mathfrak{X}(\pi(M))$ [9].

Concerning the Berwald connection on the pullback bundle, we have

Theorem 2.2. [9] *Let (M, L) be a Finsler manifold. There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

- a) $D^\circ L = 0$;
- b) $D^{h^{\circ X}}$ is torsion-free: $T^\circ = 0$;
- c) The (v)hv-torsion tensor P° of D° vanishes:

$$\hat{P}^\circ(\bar{X}, \bar{Y}) = 0.$$

Such a connection is called the Berwald connection associated with the Finsler manifold (M, L) .

Theorem 2.3. [9] *Let (M, L) be a Finsler manifold. The Berwald connection D° is expressed in terms of the Cartan connection ∇ as*

$$D_x^\circ \bar{Y} = \nabla_x \bar{Y} + \hat{P}(\rho X, \bar{Y}) - T(KX, \bar{Y}), \\ \forall X \in \mathfrak{X}(TM), \bar{Y} \in \mathfrak{X}(\pi(M)).$$

In particular, we have:

- a) $D_{\gamma\bar{X}}^\circ \bar{Y} = \nabla_{\gamma\bar{X}} \bar{Y} - T(\bar{X}, \bar{Y})$,
- b) $D_{\beta\bar{X}}^\circ \bar{Y} = \nabla_{\beta\bar{X}} \bar{Y} + \hat{P}(\bar{X}, \bar{Y})$.

Finally, for a Finsler manifold (M, L) , we use the following definitions and notations:

$$\ell(\bar{X}) := L^{-1}g(\bar{X}, \bar{\eta}),$$

$$h := g - \ell \otimes \ell: \text{ The angular metric tensor,}$$

$$T(\bar{X}, \bar{Y}, \bar{Z}) := g(T(\bar{X}, \bar{Y}), \bar{Z}): \text{ The Cartan tensor,}$$

$$C(\bar{X}) := Tr\{\bar{Y} \mapsto T(\bar{X}, \bar{Y})\}: \text{ The contracted torsion,}$$

$$g(\bar{C}, \bar{X}) := C(\bar{X}), \bar{C} \text{ is the } \pi\text{-vector field associated with the } \pi\text{-form } C,$$

$$S \text{ (resp. } P, R): \text{ The v-curvature (hv-curvature, h-curvature) tensor of Cartan connection.}$$

$$Ric^v(\bar{X}, \bar{Y}) := Tr\{\bar{Z} \mapsto S(\bar{X}, \bar{Z})\bar{Y}\}: \text{ The vertical Ricci tensor,}$$

$$g(Ric_0^v(\bar{X}), \bar{Y}) := Ric^v(\bar{X}, \bar{Y}): \text{ The vertical Ricci map}$$

$$Ric_0^v, \\ Sc^v := Tr\{\bar{X} \mapsto Ric_0^v(\bar{X})\}: \text{ The vertical Scalar curvature,}$$

$$\overset{h}{\nabla}: \text{ the h-covariant derivative associated with the Cartan connection,}$$

$$\overset{v}{\nabla}: \text{ the v-covariant derivative associated with the Cartan connection.}$$

3. Concircular π -Vector Fields on a Finsler Manifold

The notion of a concircular vector field has been studied in Riemannian geometry by Adati and Miyazawa [5]. The notion of a concurrent vector field has been investigated locally (resp. *intrinsically*) in Finsler geometry by

Matsumoto and Eguchi [3], Tachibana [2] (resp. Youssef *et al.* [4]). In this section, we investigate *intrinsically* the notion of a concircular π -vector field in Finsler geometry, which generalizes the concept of a concircular vector field in Riemannian geometry and the concept of concurrent vector field in Finsler geometry.

Definition 3.1. Let (M, L) be a Finsler manifold. A π -vector field $\bar{\zeta}(x, y) \in \mathfrak{X}(\pi(M))$ is called a *concircular π -vector field* (with respect to the Cartan connection) if it satisfies the following conditions:

$$\nabla_{\beta\bar{X}}\bar{\zeta} = \alpha(\bar{X})\bar{\zeta} + \psi(x)\bar{X}, \tag{1}$$

$$\nabla_{\gamma\bar{X}}\bar{\zeta} = 0, \tag{2}$$

where $\alpha(\bar{X}) := d\sigma(\beta\bar{X})$; $\sigma(x)$ and $\psi(x)$ are two non-zero scalar functions on TM .

In particular, if $\sigma(x)$ is constant and $\psi(x) = -1$, then $\bar{\zeta}$ is a concurrent π -vector field.

The following two Lemmas are useful for subsequent use.

Lemma 3.2. Let (M, L) be a Finsler manifold. If $\bar{\zeta} \in \mathfrak{X}(\pi(M))$ is a concircular π -vector field and $\omega \in \mathfrak{X}^*(\pi(M))$ is the π -form defined by $\omega := i_{\bar{\zeta}}g$, then ω has the properties:

- a) $(\nabla_{\beta\bar{X}}\omega)(\bar{Y}) = \alpha(\bar{X})\omega(\bar{Y}) + \psi(x)g(\bar{X}, \bar{Y})$,
- b) $(\nabla_{\gamma\bar{X}}\omega)(\bar{Y}) = 0$.

Proof.

a) Using the fact that $\nabla g = 0$, we have

$$\begin{aligned} (\nabla_{\beta\bar{X}}\omega)(\bar{Y}) &= \nabla_{\beta\bar{X}}g(\bar{\zeta}, \bar{Y}) - g(\bar{\zeta}, \nabla_{\beta\bar{X}}\bar{Y}) \\ &= (\nabla_{\beta\bar{X}}g)(\bar{\zeta}, \bar{Y}) + g(\nabla_{\beta\bar{X}}\bar{\zeta}, \bar{Y}) \\ &= g(\alpha(\bar{X})\bar{\zeta} + \psi(x)\bar{X}, \bar{Y}); \end{aligned}$$

b) The proof is similar to that of (a). \square

Lemma 3.3. Let (M, L) be a Finsler manifold and D° the Berwald connection on $\pi^{-1}(TM)$. Then, we have

- a) A π -vector field $\bar{Y} \in \mathfrak{X}(\pi(M))$ is independent of the directional argument y if, and only if, $D_{\gamma\bar{X}}^\circ\bar{Y} = 0$ for all $\bar{X} \in \mathfrak{X}(\pi(M))$;
- b) A scalar (vector) π -form A is independent of the directional argument y if, and only if, $D_{\gamma\bar{X}}^\circ A = 0$ for

- e) $P(\bar{X}, \bar{Y})\bar{\zeta} = \psi(x)T(\bar{X}, \bar{Y})$, $P(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = -\psi(x)T(\bar{X}, \bar{Y}, \bar{Z})$;
- f) $(\nabla_{\gamma\bar{Z}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) = \psi(x)(\nabla_{\gamma\bar{Z}}T)(\bar{X}, \bar{Y})$;
- g) $(\nabla_{\beta\bar{Z}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) = (\mu(\bar{Z}) - \psi(x)\alpha(\bar{Z}))T(\bar{X}, \bar{Y}) + \psi(x)(\nabla_{\beta\bar{Z}}T)(\bar{X}, \bar{Y}) - \psi(x)P(\bar{X}, \bar{Y})\bar{Z}$;
- h) $(\nabla_{\beta\bar{\zeta}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) = (\mu(\bar{\zeta}) - \psi(x)\alpha(\bar{\zeta}) - \psi^2(x))T(\bar{X}, \bar{Y}) + \psi(x)(\nabla_{\beta\bar{\zeta}}T)(\bar{X}, \bar{Y})$;

For the h-curvature tensor R , the following relations hold¹:

¹ $\mathfrak{A}_{\bar{X}, \bar{Y}}\{A(\bar{X}, \bar{Y})\}$ denotes the alternate sum $A(\bar{X}, \bar{Y}) - A(\bar{Y}, \bar{X})$.

all $\bar{X} \in \mathfrak{X}(\pi(M))$.

Proof. We prove (a) only; the proof of (b) is similar. Let $\bar{X} = X^i\bar{\partial}_i, \bar{Y} = Y^j\bar{\partial}_j$. Then,

$$\begin{aligned} D_{\gamma\bar{X}}^\circ\bar{Y} &= \nabla_{\gamma\bar{X}}\bar{Y} - T(\bar{X}, \bar{Y}) = \rho[\gamma\bar{X}, \beta\bar{Y}] \\ &= \rho[X^i\gamma(\bar{\partial}_i), Y^j\beta(\bar{\partial}_j)] = \rho[X^i\dot{\partial}_i, Y^j\delta_j] \\ &= X^iY^j\rho[\dot{\partial}_i, \delta_j] + X^i(\dot{\partial}_iY^j)\rho(\delta_j) \\ &\quad - Y^j\delta_j(X^i)\rho(\dot{\partial}_i), \end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x^i}, \dot{\partial}_i = \frac{\partial}{\partial y^i}$ and $\delta_i, \bar{\partial}_i$ are respectively

the bases of the horizontal space and the pullback fibre.

As $\rho(\partial_i) = \bar{\partial}_i, \rho(\dot{\partial}_i) = 0, \rho(\delta_i) = \bar{\partial}_i$, we have $D_{\gamma\bar{X}}^\circ\bar{Y} = X^i(\dot{\partial}_iY^j)\bar{\partial}_j$, and so

$$\begin{aligned} D_{\gamma\bar{X}}^\circ\bar{Y} = 0, \forall \bar{X} &\Leftrightarrow X^i(\dot{\partial}_iY^j)\bar{\partial}_j = 0, \\ \forall X^i &\Leftrightarrow \dot{\partial}_iY^j = 0, \\ \forall i, j &\Leftrightarrow \bar{Y} \end{aligned}$$

is independent of y . \square

Remark 3.4. From Definition 2.1, Lemma 2.3 and Theorem 1.3, we conclude that

- a) $d\psi(\gamma\bar{X}) = D_{\gamma\bar{X}}^\circ\psi(x) = \nabla_{\gamma\bar{X}}\psi(x) = 0$;
- b) $(D_{\gamma\bar{X}}^\circ\alpha)(\bar{Y}) = (\nabla_{\gamma\bar{X}}\alpha)(\bar{Y}) + \alpha(T(\bar{X}, \bar{Y})) = 0$;
- c) $(D_{\gamma\bar{X}}^\circ\mu)(\bar{Y}) = (\nabla_{\gamma\bar{X}}\mu)(\bar{Y}) + \mu(T(\bar{X}, \bar{Y})) = 0$,

where $\mu(\bar{X}) := d\psi(\beta\bar{X})$.

Now, we have the following

Theorem 3.5. Let $\zeta \in \mathfrak{X}(\pi(M))$ be a concircular π -vector field on (M, L) .

For the ν -curvature tensor S , the following relations hold¹:

- a) $S(\bar{X}, \bar{Y})\bar{\zeta} = 0, S(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = 0$;
- b) $(\nabla_{\gamma\bar{Z}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0$;
- c) $(\nabla_{\beta\bar{Z}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = -\psi(x)S(\bar{X}, \bar{Y})\bar{Z}$;
- d) $(\nabla_{\beta\bar{\zeta}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0$.

For the hv-curvature tensor P , the following relations hold:

- i) $R(\bar{X}, \bar{Y})\bar{\zeta} = \mathfrak{A}_{\bar{X}, \bar{Y}} \{ (\mu(\bar{Y}) - \psi(x)\alpha(\bar{Y}))\bar{X} \};$
- j) $R(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = \mathfrak{A}_{\bar{X}, \bar{Y}} \{ (\mu(\bar{X}) - \psi(x)\alpha(\bar{X}))g(\bar{Y}, \bar{Z}) \};$
- k) $(\nabla_{\gamma\bar{Z}}R)(\bar{X}, \bar{Y}, \bar{\zeta}) = \mathfrak{A}_{\bar{X}, \bar{Y}} \{ [\mu(T(\bar{Z}, \bar{Y})) - \alpha(T(\bar{Z}, \bar{Y}))]\bar{X} \};$
- l) $(\nabla_{\beta\bar{Z}}R)(\bar{X}, \bar{Y}, \bar{\zeta})$
 $= \mathfrak{A}_{\bar{X}, \bar{Y}} \{ ((\nabla_{\beta\bar{Z}}\mu)(\bar{Y}) - \psi(x)(\nabla_{\beta\bar{Z}}\alpha)(\bar{Y}) + \psi(x)\alpha(\bar{Z})\alpha(\bar{Y}))\bar{X} \}$
 $- \mathfrak{A}_{\bar{X}, \bar{Y}} \{ (\mu(\bar{Z})\alpha(\bar{Y}) + \alpha(\bar{Z})\mu(\bar{Y}))\bar{X} \} - \psi(x)R(\bar{X}, \bar{Y})\bar{Z};$
- m) $(\nabla_{\beta\bar{\zeta}}R)(\bar{X}, \bar{Y}, \bar{\zeta})$
 $= \mathfrak{A}_{\bar{X}, \bar{Y}} \{ \psi(x)\alpha(\bar{\zeta})\alpha(\bar{Y}) - \psi(x)\mu(\bar{Y}) + \psi^2(x)\alpha(\bar{Y})\bar{X} \}$
 $+ \mathfrak{A}_{\bar{X}, \bar{Y}} \{ ((\nabla_{\beta\bar{\zeta}}\mu)(\bar{Y}) - \psi(x)(\nabla_{\beta\bar{\zeta}}\alpha)(\bar{Y}))\bar{X} \} - \mathfrak{A}_{\bar{X}, \bar{Y}} \{ (\mu(\bar{\zeta})\alpha(\bar{Y}) + \alpha(\bar{\zeta})\mu(\bar{Y}))\bar{X} \}.$

Proof. The proof follows from the properties of the curvature tensors S, P and R , investigated in [10], together with Definition 2.1 and Remark 2.4, taking into account the fact that the (h)h-torsion of the Cartan connection vanishes. \square

In view of the above theorem, we retrieve a result of [4] concerning concurrent π -vector fields.

Corollary 3.6. *Let $\bar{\zeta} \in \mathfrak{X}(\pi(M))$ be a concurrent π -vector field on (M, L) .*

For the v-curvature tensor S , the following relations hold:

- a) $S(\bar{X}, \bar{Y})\bar{\zeta} = 0, S(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = 0;$
- b) $(\nabla_{\gamma\bar{Z}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0,$
 $(\nabla_{\beta\bar{Z}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = S(\bar{X}, \bar{Y})\bar{Z};$
- c) $(\nabla_{\beta\bar{\zeta}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0.$

For the hv-curvature tensor P , the following relations hold:

- d) $P(\bar{X}, \bar{Y})\bar{\zeta} = -T(\bar{Y}, \bar{X}),$
 $P(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = T(\bar{X}, \bar{Y}, \bar{Z});$
- e) $(\nabla_{\gamma\bar{Z}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) = -(\nabla_{\gamma\bar{Z}}T)(\bar{Y}, \bar{X}),$
 $(\nabla_{\beta\bar{Z}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) = -(\nabla_{\beta\bar{Z}}T)(\bar{Y}, \bar{X}) + P(\bar{X}, \bar{Y})\bar{Z};$
- f) $(\nabla_{\beta\bar{\zeta}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) = -(\nabla_{\beta\bar{\zeta}}T)(\bar{Y}, \bar{X}) - T(\bar{Y}, \bar{X}).$

For the h-curvature tensor R , the following relations hold:

- g) $R(\bar{X}, \bar{Y})\bar{\zeta} = 0, R(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = 0;$
- h) $(\nabla_{\gamma\bar{Z}}R)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0,$
 $(\nabla_{\beta\bar{Z}}R)(\bar{X}, \bar{Y}, \bar{\zeta}) = R(\bar{X}, \bar{Y})\bar{Z};$

i) $(\nabla_{\beta\bar{\zeta}}R)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0.$

Proof. The proof follows from Theorem 2.5 by letting $\sigma(x)$ be a constant function on M and $\psi(x) = -1$. \square

Proposition 3.7. *Let $\bar{\zeta}$ be a concircular π -vector field. For every $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, we have:*

- a) $T(\bar{X}, \bar{\zeta}) = T(\bar{\zeta}, \bar{X}) = 0;$
- b) $\hat{P}(\bar{X}, \bar{\zeta}) = \hat{P}(\bar{\zeta}, \bar{X}) = 0;$
- c) $\hat{R}(\bar{X}, \bar{\zeta}) = K[\beta\bar{X}, \beta\bar{\zeta}];$
- d) $P(\bar{X}, \bar{\zeta})\bar{Y} = P(\bar{\zeta}, \bar{X})\bar{Y} = 0;$
- e) $\mathfrak{A}_{\bar{X}, \bar{Y}} \{ (\mu(\bar{Y}) - \psi(x)\alpha(\bar{Y}))\omega(\bar{X}) \} = 0;$
- f) $\mu(T(\bar{X}, \bar{Y})) = \psi(x)\alpha(T(\bar{X}, \bar{Y})).$

Proof.

a) From Theorem 2.5(e), by setting $\bar{Z} = \bar{\zeta}$ and making use of the symmetry of T and the identity $g(P(\bar{X}, \bar{Y})\bar{Z}, \bar{Z}) = 0$ [10], we obtain

$$0 = g(P(\bar{X}, \bar{Y})\bar{\zeta}, \bar{\zeta}) = -\psi(x)T(\bar{X}, \bar{Y}, \bar{\zeta}) = -\psi(x)g(T(\bar{X}, \bar{\zeta}), \bar{Y}).$$

From which, since $\psi(x) \neq 0$, the result follows.

b) We have [10]

$$\hat{P}(\bar{X}, \bar{Y}) = (\nabla_{\beta\bar{\eta}}T)(\bar{X}, \bar{Y}).$$

From which, setting $\bar{X} = \bar{\zeta}$, it follows that

$$\begin{aligned} &\hat{P}(\bar{\zeta}, \bar{Y}) \\ &= (\nabla_{\beta\bar{\eta}}T)(\bar{\zeta}, \bar{Y}) \\ &= \nabla_{\beta\bar{\eta}}T(\bar{\zeta}, \bar{Y}) - T(\nabla_{\beta\bar{\eta}}\bar{\zeta}, \bar{Y}) - T(\bar{\zeta}, \nabla_{\beta\bar{\eta}}\bar{Y}) \\ &= \nabla_{\beta\bar{\eta}}T(\bar{\zeta}, \bar{Y}) - \alpha(\bar{\eta})T(\bar{\zeta}, \bar{Y}) \\ &\quad - \psi(x)T(\bar{\eta}, \bar{Y}) - T(\bar{\zeta}, \nabla_{\beta\bar{\eta}}\bar{Y}). \end{aligned}$$

Hence, making use of (a), the symmetry of \hat{P} and the fact that $T(\bar{X}, \bar{\eta}) = 0$, the result follows.

c) Clear.

d) We have from [10],

$$\begin{aligned}
 &P(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \\
 &= g\left(\left(\nabla_{\beta\bar{Z}}T\right)(\bar{Y}, \bar{X}), \bar{W}\right) - g\left(\left(\nabla_{\beta\bar{W}}T\right)(\bar{Y}, \bar{X}), \bar{Z}\right) \quad (3) \\
 &- g\left(T(\bar{X}, \bar{W}), \hat{P}(\bar{Z}, \bar{Y})\right) + g\left(T(\bar{X}, \bar{Z}), \hat{P}(\bar{W}, \bar{Y})\right).
 \end{aligned}$$

From which, by setting $\bar{Y} = \bar{\zeta}$ in (3), using (b) and the symmetry of T , we conclude that $P(\bar{X}, \bar{\zeta})\bar{Z} = 0$. Similarly, setting $\bar{X} = \bar{\zeta}$ in (3), using (a) and the symmetry of T , we get $P(\bar{\zeta}, \bar{Y})\bar{Z} = 0$.

e) The proof follows from Theorem 2.5 (j) by setting $\bar{Z} = \bar{\zeta}$, taking into account the fact that $g(R(\bar{X}, \bar{Y})\bar{Z}, \bar{Z}) = 0$ [10].

f) We have

$$\mathfrak{A}_{\bar{X}, \bar{Y}}\left\{\left(\mu(\bar{Y}) - \psi(x)\alpha(\bar{Y})\right)\omega(\bar{X})\right\} = 0.$$

Hence, there exists a scalar function λ such that

$$\mu(\bar{X}) - \psi(x)\alpha(\bar{X}) = \lambda\omega(\bar{X}).$$

Consequently, using (a) and the symmetry of T , we get

$$\begin{aligned}
 &\mu\left(T(\bar{X}, \bar{Y})\right) - \psi(x)\alpha\left(T(\bar{X}, \bar{Y})\right) \\
 &= \lambda\omega\left(T(\bar{X}, \bar{Y})\right) = g\left(\left(T(\bar{X}, \bar{Y})\right), \bar{\zeta}\right) = 0.
 \end{aligned}$$

This completes the proof. \square

Theorem 3.8. *A concircular π -vector field $\bar{\zeta}$ and its associated π -form ω are independent of the directional argument y .*

Proof. By Theorem 1.3(a), we have

$$D_{\gamma\bar{X}}^\circ\bar{Y} = \nabla_{\gamma\bar{X}}\bar{Y} - T(\bar{X}, \bar{Y}).$$

From which, by setting $\bar{Y} = \bar{\zeta}$ and taking into account (2), Proposition 2.7(a) and Lemma 2.3, we conclude that $D_{\gamma\bar{X}}^\circ\bar{\zeta} = 0$ and $\bar{\zeta}$ is thus independent of the directional argument y .

On the other hand, we have from the above relation

$$\left(D_{\gamma\bar{X}}^\circ\omega\right)(\bar{Y}) = \left(\nabla_{\gamma\bar{X}}\omega\right)(\bar{Y}) + T(\bar{X}, \bar{Y}, \bar{\zeta}).$$

This, together with Lemma 2.2(b), Proposition 2.7(a) and the symmetry of T , imply that ω is also independent of the directional argument y . \square

In view of Theorem 1.3 and Proposition 2.7, we have

Theorem 3.9. *A π -vector field $\bar{\zeta}$ on (M, L) is concircular with respect to Cartan connection if, and only if, it is concircular with respect to Berwald connection.*

Remark 3.10. *As a consequence of the above results, we retrieve a result of [4] concerning concurrent π -*

vector fields: A concurrent π -vector field $\bar{\zeta}$ and its associated π -form ω are independent of the directional argument y . Moreover, a π -vector field $\bar{\zeta}$ on (M, L) is concurrent with respect to Cartan connection if, and only if, it is concurrent with respect to Berwald connection.

4. Special Finsler Spaces Admitting Concircular π -Vector Fields

Special Finsler manifolds arise by imposing extra conditions on the curvature and torsion tensors available in the space. Due to the abundance of such geometric objects in the context of Finsler geometry, special Finsler spaces are quite numerous. The study of these spaces constitutes a substantial part of research in Finsler geometry. A complete and systematic study of special Finsler spaces, from a global point of view, has been accomplished in [7].

In this section, we investigate the effect of the existence of a concircular π -vector field on some important special Finsler spaces. The intrinsic definitions of the special Finsler spaces treated here are quoted from [7].

For later use, we need the following lemma.

Lemma 4.1. *Let (M, L) be a Finsler manifold admitting a concircular π -vector field $\bar{\zeta}$. Then, we have:*

a) The concircular π -vector field $\bar{\zeta}$ is everywhere non-zero.

b) The scalar function $B := g(\bar{\zeta}, \bar{\eta})$ is everywhere non-zero.

c) The π -vector field $\bar{m} := \bar{\zeta} - \frac{B}{L^2}\bar{\eta}$ is everywhere non-zero and is orthogonal to $\bar{\eta}$.

d) The π -vector fields \bar{m} and $\bar{\zeta}$ satisfy $g(\bar{m}, \bar{\zeta}) = g(\bar{m}, \bar{m}) \neq 0$.

e) The scalar function $h(\bar{\zeta}, \bar{\zeta})$ is everywhere non-zero.

Proof.

a) Follows by Definition 2.1.

b) Suppose that $B := g(\bar{\zeta}, \bar{\eta}) = 0$, then

$$\begin{aligned}
 0 &= \left(\nabla_{\gamma\bar{X}}g\right)(\bar{\zeta}, \bar{\eta}) = \nabla_{\gamma\bar{X}}g(\bar{\zeta}, \bar{\eta}) - g(\bar{\zeta}, \bar{X}) \\
 &= -g(\bar{\zeta}, \bar{X}), \forall \bar{X} \in \mathfrak{X}(\pi(M)).
 \end{aligned}$$

Hence, as g is nondegenerate, $\bar{\zeta}$ vanishes, which contradicts (a). Consequently, $B \neq 0$.

c) If $\bar{m} = 0$, then $L^2\bar{\zeta} - B\bar{\eta} = 0$. Differentiating covariantly with respect to $\gamma\bar{X}$, we get

$$2g(\bar{X}, \bar{\eta})\bar{\zeta} - B\bar{X} - g(\bar{X}, \bar{\zeta})\bar{\eta} = 0. \quad (1)$$

From which,

$$g(\bar{X}, \bar{\zeta}) = \frac{B}{L^2}g(\bar{X}, \bar{\eta}). \quad (2)$$

By (1), using (2), we obtain

$$\begin{aligned} 0 &= 2g(\bar{X}, \bar{\eta})g(\bar{Y}, \bar{\zeta}) - Bg(\bar{X}, \bar{Y}) - g(\bar{X}, \bar{\zeta})g(\bar{Y}, \bar{\eta}) \\ &= 2\frac{B}{L^2}g(\bar{Y}, \bar{\eta})g(\bar{X}, \bar{\eta}) - Bg(\bar{X}, \bar{Y}) \\ &\quad - \frac{B}{L^2}g(\bar{X}, \bar{\eta})g(\bar{Y}, \bar{\eta}) \\ &= -B\left\{g(\bar{X}, \bar{Y}) - \frac{1}{L^2}g(\bar{Y}, \bar{\eta})g(\bar{X}, \bar{\eta})\right\} \\ &= -Bh(\bar{X}, \bar{Y}). \end{aligned}$$

From which, since $B \neq 0$, we are led to a contradiction: $h = 0$. Consequently, $\bar{m} \neq 0$.

On the other hand, the orthogonality of the two π -vector fields \bar{m} and $\bar{\eta}$ follows from the identities $g(\bar{\eta}, \bar{\eta}) = L^2$ and $g(\bar{\eta}, \bar{\zeta}) = B$.

d) Follows from (c).

e) Follows from (d), (c) and the fact that

$$h(X, \eta) = h(\eta, X) = 0. \quad \square$$

Definition 4.2. A Finsler manifold (M, L) is said to be:

a) Riemannian if the metric tensor $g(x, y)$ is independent of y or, equivalently, if $T = 0$;

b) Berwald if the torsion tensor T is horizontally parallel: $\nabla T = 0$;

c) Landsberg if the $\nu(h\nu)$ -torsion tensor $\hat{P} = 0$ or, equivalently, if $\nabla_{\beta\bar{\eta}}T = 0$.

Theorem 4.3. A Landsberg manifold admitting a concircular π -vector field $\bar{\zeta}$ is Riemannian.

Proof. Suppose that (M, L) is Landsberg, then $\hat{P} = 0$. Consequently, the $h\nu$ -curvature P vanishes [10]. Hence, by Theorem 2.5(e),

$$0 = P(\bar{X}, \bar{Y}, \bar{Z}, \bar{\zeta}) = -\psi(x)T(\bar{X}, \bar{Y}, \bar{Z}).$$

From which, taking into account the fact that $\psi(x)$ is a non-zero function, it follows that $T = 0$. Hence the result follows. \square

As a consequence of the above result, we get

Corollary 4.4. The existence of a concircular π -vector field $\bar{\zeta}$ implies that the three notions of being Landsberg, Berwald and Riemannian coincide.

Definition 4.5. A Finsler manifold (M, L) is said to be:

a) C_2 -like if $\dim M \geq 2$ and the Cartan tensor T has the form

$$T(\bar{X}, \bar{Y}, \bar{Z}) = \frac{1}{C^2}C(\bar{X})C(\bar{Y})C(\bar{Z}).$$

b) C -reducible if $\dim M \geq 3$ and the Cartan tensor T has the form²

² $\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}$ denotes the cyclic sum over the arguments \bar{X}, \bar{Y} and \bar{Z} .

$$T(\bar{X}, \bar{Y}, \bar{Z}) = \frac{1}{n+1}\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}\{h(\bar{X}, \bar{Y})C(\bar{Z})\}. \quad (3)$$

c) semi- C -reducible if $\dim M \geq 3$ and the Cartan tensor T has the form

$$\begin{aligned} T(\bar{X}, \bar{Y}, \bar{Z}) &= \frac{\mu}{n+1}\mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}\{h(\bar{X}, \bar{Y})C(\bar{Z})\} + \frac{\tau}{C^2}C(\bar{X})C(\bar{Y})C(\bar{Z}), \end{aligned} \quad (4)$$

where $C^2 := C(\bar{C}) \neq 0$, μ and τ are scalar functions satisfying $\mu + \tau = 1$.

d) quasi- C -reducible if $\dim M \geq 3$ and the Cartan tensor T has the form

$$T(\bar{X}, \bar{Y}, \bar{Z}) = \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}}\{A(\bar{X}, \bar{Y})C(\bar{Z})\},$$

where A is a symmetric π -tensor field satisfying $A(\bar{X}, \bar{\eta}) = 0$.

Theorem 4.6. Let (M, L) be a Finsler manifold ($\dim M \geq 3$) admitting a concircular π -vector field $\bar{\zeta}$.

a) If (M, L) is quasi- C -reducible, then it is Riemannian, provided that $A(\bar{\zeta}, \bar{\zeta}) \neq 0$.

b) If (M, L) is C -reducible, then it is Riemannian.

c) If (M, L) is semi- C -reducible, then it is C_2 -like.

Proof.

a) Follows from the defining property of quasi- C -reducibility by setting $\bar{X} = \bar{Y} = \bar{\zeta}$ and using the fact that $C(\bar{\zeta}) = 0$ and the given assumption $A(\bar{\zeta}, \bar{\zeta}) \neq 0$;

b) Setting $\bar{X} = \bar{Y} = \bar{\zeta}$ in (3.3), taking into account Proposition 2.7(a), Lemma 3.1(e) and $C(\bar{\zeta}) = 0$, it follows that $C = 0$, which is equivalent to $T = 0$ (Deicke theorem [11]);

c) Let (M, L) be semi- C -reducible. Setting $\bar{X} = \bar{Y} = \bar{\zeta}$ and $\bar{Z} = \bar{C}$ in (3.4), taking into account Proposition 2.7(a) and $C(\bar{\zeta}) = 0$, we get

$$\mu h(\bar{\zeta}, \bar{\zeta})C(\bar{C}) = 0.$$

From which, since $h(\bar{\zeta}, \bar{\zeta}) \neq 0$ (Lemma 3.1(e)) and $C(\bar{C}) \neq 0$, it follows that $\mu = 0$.

Consequently, (M, L) is C_2 -like. \square

Definition 4.7. A Finsler manifold (M, L) is said to be S_3 -like if $\dim M \geq 4$ and the ν -curvature tensor S has the form:

$$\begin{aligned} S(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= \frac{Sc^\nu}{(n-1)(n-2)}\{h(\bar{X}, \bar{Z})h(\bar{Y}, \bar{W}) - h(\bar{X}, \bar{W})h(\bar{Y}, \bar{Z})\}. \end{aligned} \quad (5)$$

Theorem 4.8. If an S_3 -like manifold admits a concircular π -vector field $\bar{\zeta}$, then the ν -curvature tensor S vanishes.

Proof. Setting $\bar{X} = \bar{Z} = \bar{\zeta}$ in (3.5), taking Theorem 2.5 into account, we immediately get

$$\frac{Sc^v}{(n-1)(n-2)} \{ \hbar(\bar{\zeta}, \bar{\zeta}) \hbar(\bar{Y}, \bar{W}) - \hbar(\bar{\zeta}, \bar{W}) \hbar(\bar{Y}, \bar{\zeta}) \} = 0$$

Taking the trace of the above equation, we have

$$\frac{Sc^v}{(n-1)(n-2)} \{ (n-1) \hbar(\bar{\zeta}, \bar{\zeta}) - \hbar(\bar{\zeta}, \bar{\zeta}) \} = 0$$

Consequently,

$$\frac{Sc^v}{(n-1)} \hbar(\bar{\zeta}, \bar{\zeta}) = 0$$

From which, since $\hbar(\bar{\zeta}, \bar{\zeta}) \neq 0$ (Lemma 3.1(e)), the vertical scalar curvature Sc^v vanishes. Now, again, from (3.5), the result follows. \square

Definition 4.9. A Finsler manifold (M, L) , where $\dim M \geq 3$, is said to be:

a) P_2 -like if the hv-curvature tensor P has the form:

$$P(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \varphi(\bar{Z})T(\bar{X}, \bar{Y}, \bar{W}) - \varphi(\bar{W})T(\bar{X}, \bar{Y}, \bar{Z}), \tag{6}$$

where φ is a $a(1)\pi$ -form, positively homogeneous of degree 0.

b) P -reducible if the π -tensor field

$$\hat{P}(\bar{X}, \bar{Y}, \bar{Z}) := g(\hat{P}(\bar{X}, \bar{Y}), \bar{Z}) \text{ has the form}$$

$$\hat{P}(\bar{X}, \bar{Y}, \bar{Z}) = \delta(\bar{X})\hbar(\bar{Y}, \bar{Z}) + \delta(\bar{Y})\hbar(\bar{X}, \bar{Z}) + \delta(\bar{Z})\hbar(\bar{X}, \bar{Y}), \tag{7}$$

where δ is the $a(1)\pi$ -form defined by

$$\delta(\bar{X}) = \frac{1}{n+1} (\nabla_{\beta\bar{\eta}} C)(\bar{X}).$$

Theorem 4.10. Let (M, L) be a Finsler manifold ($\dim M \geq 3$) admitting a concircular π -vector field $\bar{\zeta}$.

a) If (M, L) is P_2 -like, then it is Riemannian, provided that $\varphi(\bar{\zeta}) \neq \psi(x)$.

b) If (M, L) is P -reducible, then it is Landsbergian.

Proof.

a) Setting $\bar{Z} = \bar{\zeta}$ in (3.6), taking into account Theorem 2.5 and Proposition 2.7, we immediately get

$$(\varphi(\bar{\zeta}) - \psi(x))T(\bar{X}, \bar{Y}) = 0.$$

Hence, the result follows.

b) Setting $\bar{X} = \bar{Y} = \bar{\zeta}$ in (3.7) and using the identity $(\nabla_{\beta\bar{\eta}} C)(\bar{\zeta}) = 0$, we conclude that $\hbar(\bar{\zeta}, \bar{\zeta})(\nabla_{\beta\bar{\eta}} C)(\bar{Z}) = 0$, with $\hbar(\bar{\zeta}, \bar{\zeta}) \neq 0$ (Lemma 3.1(e)). Consequently, $\nabla_{\beta\bar{\eta}} C = 0$. Hence, again, from Definition 3.9(b), the (v)hv-torsion tensor $\hat{P} = 0$. \square

Definition 4.11. A Finsler manifold (M, L) of $\dim M \geq 3$ is said to be h -isotropic if there exists a

scalar function k_o such that the horizontal curvature tensor R has the form

$$R(\bar{X}, \bar{Y})\bar{Z} = k_o \{ g(\bar{X}, \bar{Z})\bar{Y} - g(\bar{Y}, \bar{Z})\bar{X} \},$$

where k_o is called the scalar curvature.

Theorem 4.12. For an h -isotropic Finsler manifold admitting a concircular π -vector field $\bar{\zeta}$, the scalar curvature k_o is given by

$$k_o = -\frac{A(\bar{m})}{g(\bar{m}, \bar{\zeta})},$$

where $A := \mu - \psi(x)\alpha$.

Proof. From Definition 3.11, by setting $\bar{Z} = \bar{\zeta}$ and $\bar{X} = \bar{m}$, we have

$$R(\bar{m}, \bar{Y})\bar{\zeta} = k_o \{ g(\bar{X}, \bar{\zeta})\bar{Y} - g(\bar{Y}, \bar{\zeta})\bar{m} \}. \tag{8}$$

On the other hand, using Theorem 2.5(i), we have

$$R(\bar{m}, \bar{Y})\bar{\zeta} = A(\bar{Y})\bar{m} - A(\bar{m})\bar{Y}, \tag{9}$$

From (8) and (9), it follows that

$$k_o \{ g(\bar{m}, \bar{\zeta})\bar{Y} - g(\bar{Y}, \bar{\zeta})\bar{m} \} = A(\bar{Y})\bar{m} - A(\bar{m})\bar{Y}.$$

Taking the trace of the above equation, we get

$$k_o (n-1) g(\bar{m}, \bar{\zeta}) = (1-n) A(\bar{m}).$$

Hence, the scalar k_o is given by

$$k_o = -\frac{A(\bar{m})}{g(\bar{m}, \bar{\zeta})}. \tag{10}$$

This completes the proof. \square

Corollary 4.13. For an h -isotropic Finsler manifold admitting a concurrent π -vector field $\bar{\zeta}$, the h -curvature R vanishes.

Proof. If $\bar{\zeta}$ is concurrent, then the π -form A vanishes. Hence, using (10), the scalar k_o vanishes. Consequently, from Definition 3.11, the h -curvature R vanishes. \square

5. Different Types of Recurrent Finsler Manifolds Admitting Concircular π -Vector Fields

In this section, we investigate intrinsically the effect of the existence of a concircular π -vector field on recurrent Finsler manifolds. We study different types of recurrence (with respect to Cartan connection).

Let us begin with the first type of recurrence related to the Cartan tensor T .

Definition 5.1. A Finsler manifold (M, L) is said to be T^h -recurrent if the (h)hv-torsion tensor T has the property that

$$\overset{h}{\nabla}T = \lambda_1 \otimes T,$$

where λ_1 is a scalar (1) π -form, positively homogenous of degree zero in y , called the h -recurrence form.

Similarly, (M, L) is called T^v -recurrent if the (h)hv-torsion tensor T has the property that

$$\overset{v}{\nabla}T = \lambda_2 \otimes T,$$

where λ_2 is a scalar (1) π -form, positively homogenous of degree -1 in y , called the v -recurrence form.

Theorem 5.2. *If a T^h -recurrent Finsler manifold admits a concircular π -vector field $\bar{\zeta}$, then it is Riemannian, provided that $\lambda_1(\bar{\zeta}) \neq 0$.*

Proof. We have [10]

$$\begin{aligned} &P(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \\ &= g((\nabla_{\beta\bar{Z}}T)(\bar{Y}, \bar{X}), \bar{W}) - g((\nabla_{\beta\bar{W}}T)(\bar{Y}, \bar{X}), \bar{Z}) \\ &\quad - g(T(\bar{X}, \bar{W}), \hat{P}(\bar{Z}, \bar{Y})) + g(T(\bar{X}, \bar{Z}), \hat{P}(\bar{W}, \bar{Y})). \end{aligned}$$

Setting $\bar{W} = \bar{\zeta}$, making use of Theorem 2.5, Proposition 2.7 and the identity [10]

$$g((\nabla_{\beta\bar{Z}}T)(\bar{X}, \bar{Y}), \bar{W}) = g((\nabla_{\beta\bar{W}}T)(\bar{X}, \bar{W}), \bar{Y}),$$

we get

$$\nabla_{\beta\bar{\zeta}}T = 0.$$

On the other hand, Definition 4.1 yields

$$\nabla_{\beta\bar{\zeta}}T = \lambda_1(\bar{\zeta})T.$$

Under the given assumption, the above two equations imply that $T = 0$. Hence, (M, L) is Riemannian. \square

In view of the above theorem, we have.

Corollary 5.3. *In the presence of a concircular π -vector field $\bar{\zeta}$, the three notions of being T^h -recurrent, T^v -recurrent and Riemannian coincide, provided that $\lambda_1(\bar{\zeta}) \neq 0$.*

Proof. By Theorem 4.7 of [7], regardless of the existence of concircular π -vector fields, a T^v -recurrent Finsler space is necessarily Riemannian. On the other hand, a Riemannian space is trivially both T^h -recurrent and T^v -recurrent. \square

Remark 5.4. *Corollary 4.3 remains true if in particular a concircular π -vector field replaced by a concurrent π -vector field [4].*

The following definition gives the second type of recurrence related to the v -curvature tensor S .

Definition 5.5. *If we replace T by S in Definition 4.1, then (M, L) is said to be S^h -recurrent (S^v -recurrent).*

Theorem 5.6. *If an S^h -recurrent Finsler manifold admits a concircular π -vector field $\bar{\zeta}$, then its v -curvature tensor S vanishes.*

Proof. Suppose that (M, L) is an S^h -recurrent manifold which admits a concircular π -vector field $\bar{\zeta}$. Then, by Definition 4.5 and Theorem 2.5(a), we have

$$(\nabla_{\beta\bar{Z}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = \lambda_1(\bar{Z})S(\bar{X}, \bar{Y}, \bar{\zeta}) = 0.$$

On the other hand, by Theorem 2.5(c), we get

$$(\nabla_{\beta\bar{Z}}S)(\bar{X}, \bar{Y}, \bar{\zeta}) = -\psi(x)S(\bar{X}, \bar{Y})\bar{Z}.$$

From the above two equations, since $\psi(x) \neq 0$, the v -curvature tensor S vanishes. \square

Corollary 5.7. *Let (M, L) be a Finsler manifold which admits a concircular π -vector field. The following assertions are equivalent:*

a) (M, L) is S^h -recurrent, b) (M, L) is S^v -recurrent, c) the v -curvature tensor S vanishes.

In fact, for an S^v -recurrent Finsler manifold the v -curvature tensor S vanishes [7] regardless of the existence of concircular π -vector fields.

Remark 5.8. *We retrieve here a result of [4] concerning concurrent π -vector fields: Corollary 4.7 remains true if in particular a concircular π -vector field replaced by a concurrent π -vector field.*

In the following we give the third type of recurrence related to the hv -curvature tensor P .

Definition 5.9. *If we replace T by P in Definition 4.1, then (M, L) is said to be P^h -recurrent (P^v -recurrent).*

In view of the above definition, we have

Theorem 5.10. *Let (M, L) be a P^h -recurrent Finsler manifold admitting a concircular π -vector field $\bar{\zeta}$. Then,*

- either (a) (M, L) is Riemannian, or
- (b) (M, L) has the property that

$$(\mu - \psi(x)\alpha - \psi(x))\lambda_1(\bar{\eta}) = 0.$$

Proof. By Theorem 2.5(g), we have

$$\begin{aligned} &(\nabla_{\beta\bar{Z}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) \\ &= (\mu(\bar{Z}) - \psi(x)\alpha(\bar{Z}))T(\bar{X}, \bar{Y}) \\ &\quad + \psi(x)(\nabla_{\beta\bar{Z}}T)(\bar{X}, \bar{Y}) - \psi(x)P(\bar{X}, \bar{Y})\bar{Z}. \end{aligned} \tag{1}$$

On the other hand, by Definition 4.9 and Theorem 2.5(e), we get

$$\begin{aligned} &(\nabla_{\beta\bar{Z}}P)(\bar{X}, \bar{Y}, \bar{\zeta}) \\ &= \lambda_1(\bar{Z})P(\bar{X}, \bar{Y})\bar{\zeta} = \psi(x)\lambda_1(\bar{Z})T(\bar{X}, \bar{Y}). \end{aligned}$$

From which together with (1), it follows that

$$\begin{aligned} &\psi(x)P(\bar{X}, \bar{Y})\bar{Z} \\ &= \{\mu(\bar{Z}) - \psi(x)\alpha(\bar{Z}) - \psi(x)\lambda_1(\bar{Z})\}T(\bar{X}, \bar{Y}) \\ &\quad + \psi(x)(\nabla_{\beta\bar{Z}}T)(\bar{X}, \bar{Y}). \end{aligned}$$

By setting $\bar{Z} = \bar{\eta}$ and noting that $\hat{P}(\bar{X}, \bar{Y}) = (\nabla_{\beta\bar{\eta}} T)(\bar{X}, \bar{Y})$ [10], the above equation gives

$$\{\mu(\bar{\eta}) - \psi(x)\alpha(\bar{\eta}) - \psi(x)\lambda_1(\bar{\eta})\}T(\bar{X}, \bar{Y}) = 0.$$

Now, we have two cases: either $T = 0$ and consequently (M, L) is Riemannian, or $(\mu - \psi(x)\alpha - \psi(x)\lambda_1)(\bar{\eta}) = 0$. This completes the proof. \square

Lemma 5.11. *For a P^v -recurrent Finsler manifold, the $h\nu$ -curvature tensor P vanishes.*

Proof. Suppose that (M, L) is P^v -recurrent, then, by Definition 4.9, we get

$$(\nabla_{\gamma\bar{w}} P)(\bar{X}, \bar{\eta}, \bar{Z}) = \lambda_2(\bar{w})P(\bar{X}, \bar{\eta})\bar{Z}.$$

From which, together with the fact that $P(\bar{X}, \bar{\eta})\bar{Z} = 0$ [10] and $K \circ \gamma = id_{\mathfrak{x}(\pi(M))}$, the result follows. \square

In view of Theorem 4.10 and Lemma 4.11, we have

Theorem 5.12. *Let (M, L) be a Finsler manifold admitting a concircular π -vector field. Then, the following assertions are equivalent:*

- a) (M, L) is P^h -recurrent;
- b) (M, L) is P^v -recurrent;
- c) (M, L) is Riemannian,

provided that $(\mu - \psi(x)\alpha - \psi(x)\lambda_1)(\bar{\eta}) \neq 0$ in the P^h -recurrence case.

Remark 5.13. *In view of Theorem 4.12, we conclude that under the presence of a concurrent π -vector field $\bar{\zeta}$, the three notions of being P^h -recurrent, P^v -recurrent and Riemannian coincide, provided that $\lambda_1(\bar{\zeta}) \neq 0$.*

Finally, we focus our attention to the fourth type of recurrent Finsler manifolds related to the h -curvature tensor R .

Definition 5.14. *If we replace T by R in Definition 4.1, then (M, L) is said to be R^h -recurrent (R^v -recurrent).*

Theorem 5.15. *An R^h -recurrent Finsler manifold admitting a concircular π -vector field $\bar{\zeta}$ is h -isotropic with scalar curvature*

$$k_o = \frac{\phi_o}{n},$$

where

$$\begin{aligned} \phi_o &:= Tr(\phi), \psi(x)\phi(\bar{X}, \bar{Y}) \\ &:= \alpha(\bar{Y})A(\bar{X}) + \lambda_1(\bar{Y})A(\bar{X}) - (\nabla_{\beta\bar{Y}} A)(\bar{X}) \\ \text{and } A &:= \mu - \psi(x)\alpha. \end{aligned}$$

Moreover, if (M, L) is R^v -recurrent with $\lambda_2(\bar{\eta}) \neq 0$, then the h -curvature tensor R vanishes.

Proof. Firstly, suppose that (M, L) is an R^h -recurrent manifold which admits a concircular π -vector

field $\bar{\zeta}$. Then, by Theorem 2.5 (I), we have

$$\begin{aligned} &(\nabla_{\beta\bar{Z}} R)(\bar{X}, \bar{Y}, \bar{\zeta}) \\ &= \mathfrak{A}_{\bar{X}, \bar{Y}} \left\{ \left((\nabla_{\beta\bar{Z}} A)(\bar{Y}) - \alpha(\bar{Z})A(\bar{Y}) \right) \bar{X} \right\} \\ &\quad - \psi(x)R(\bar{X}, \bar{Y})\bar{Z}. \end{aligned}$$

On the other hand, by Definition 4.14 and Theorem 2.5(i), we get

$$\begin{aligned} &(\nabla_{\beta\bar{Z}} R)(\bar{X}, \bar{Y}, \bar{\zeta}) \\ &= \lambda_1(\bar{Z})R(\bar{X}, \bar{Y})\bar{\zeta} = \mathfrak{A}_{\bar{X}, \bar{Y}} \left\{ \lambda_1(\bar{Z})A(\bar{Y})\bar{X} \right\}. \end{aligned}$$

The above two equations imply that

$$\begin{aligned} R(\bar{X}, \bar{Y})\bar{Z} &= \frac{1}{\psi(x)} \mathfrak{A}_{\bar{X}, \bar{Y}} \\ &\left\{ \left(\alpha(\bar{Z})A(\bar{X}) + \lambda_1(\bar{Z})A(\bar{X}) - (\nabla_{\beta\bar{Z}} A)(\bar{X}) \right) \bar{Y} \right\} \\ &= \mathfrak{A}_{\bar{X}, \bar{Y}} \left\{ \phi(\bar{X}, \bar{Z})\bar{Y} \right\}. \end{aligned}$$

Consequently,

$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \phi(\bar{X}, \bar{Z})g(\bar{Y}, \bar{W}) - \phi(\bar{Y}, \bar{Z})g(\bar{X}, \bar{W}). \tag{2}$$

Hence,

$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \phi(\bar{X}, \bar{W})g(\bar{Y}, \bar{Z}) - \phi(\bar{Y}, \bar{W})g(\bar{X}, \bar{Z}).$$

From the above two relations, noting that $R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = -R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})$ [10], we get

$$\begin{aligned} &\phi(\bar{X}, \bar{Z})g(\bar{Y}, \bar{W}) - \phi(\bar{Y}, \bar{Z})g(\bar{X}, \bar{W}) \\ &+ \phi(\bar{X}, \bar{W})g(\bar{Y}, \bar{Z}) - \phi(\bar{Y}, \bar{W})g(\bar{X}, \bar{Z}) = 0 \end{aligned}$$

Taking the trace of the above relation with respect to the two arguments \bar{Y} and \bar{W} , we obtain

$$\phi(\bar{X}, \bar{Z}) = \frac{\phi_o}{n}g(\bar{X}, \bar{Z}).$$

From which, together with (4.2), we obtain

$$\begin{aligned} &R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \\ &= \frac{\phi_o}{n} \left\{ g(\bar{X}, \bar{Z})g(\bar{Y}, \bar{W}) - g(\bar{Y}, \bar{Z})g(\bar{X}, \bar{W}) \right\}. \end{aligned}$$

This means that (M, L) is h -isotropic (Definition 3.11) with scalar curvature $k_o = \frac{\phi_o}{n}$.

Finally, the second part of the theorem follows from Definition 4.14 and the identity $(\nabla_{\gamma\bar{\eta}} R)(\bar{X}, \bar{Y}, \bar{Z}) = 0$ [10]. \square

As a consequence of the above theorem, we have

Corollary 5.16. *For an R^h -recurrent Finsler manifold admitting a concurrent π -vector field $\bar{\zeta}$, the h -*

curvature tensor R vanishes.

6. Concluding Remarks

- The concept of a concircular π -vector field in Finsler geometry has been introduced and investigated from a global point of view. This generalizes, on one hand, the concept of a concircular vector field in Riemannian geometry and, on the other hand, the concept of a concurrent vector field in Finsler geometry. Various properties of concircular π -vector fields have been obtained.
- The effect of the existence of concircular π -vector fields on some of the most important special Finsler spaces has been investigated.
- Different types of recurrent Finsler manifolds admitting concircular π -vector fields have been studied.
- Almost all results of this work have been obtained in a coordinate-free form, without being trapped into the complications of indices.

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